Can you hear the shape of a drum?

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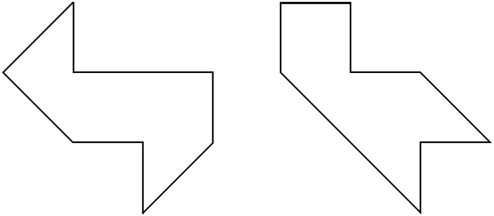
Can you hear the shape of a drum?

The question we are really asking is *"is the spectrum of the laplacian unique for all regions in* $\mathbb{R}^n$*?"*

Unfortunately no.

In 1992, Gordon, Webb, and Wolpert constructed the counterexample below.



Spectrum of an operator

For an operator L, we can look at for functions $\phi \in C_0(\Omega)$ and constants $\lambda \in \mathbb{C}$ such that:

$$L\phi = \lambda\phi$$

The spectrum of L is the set of all these 'eigenvalues' λ. When two regions have the same spectrum, we say they are isospectral.

The Wave Equation

The reason we care about this is because using the spectrum, we can construct solutions to the wave equation, which determines the sound the drum makes.

$$\Delta u = \partial_{tt} u$$

An example (1/3)

Let's start by explicitly finding solutions to the wave equation on the square $\Omega = [0,1] \times [0,1]$.

$$-\Delta u = \lambda u$$

Assume u is factorable as $u(x,y) = g(x)h(y)$ and substitute:

$$\begin{aligned} -(g''h + gh'') &= \lambda gh \\ -(\frac{g''}{g} + \frac{h''}{h}) &= \lambda \\ G(x) + H(y) &= -\lambda \end{aligned}$$

From this, we can reason that both $G(x)$ and $H(y)$ must be constants.

An example (2/3)

Since these are constant functions, we can solve for g and h with ODEs:

$$g''(x) = -c_1 g(x)$$

$$g(x) = \sin(\sqrt{c_1} x), c_1 = (n\pi)^2, n \in \mathbb{Z}$$

$$h''(y) = -c_2 h(y)$$

$$h(y) = \sin(\sqrt{c_2} y), c_2 = (m\pi)^2, m \in \mathbb{Z}$$

$$-(\frac{g''}{g} + \frac{h''}{h}) = c_1 + c_2 = \lambda$$

An example (3/3)

Now we can use this information to find a solution to the wave equation.

$$\Delta u = \partial_{tt} u$$

Make the same factoring assumption $u(x, y, t) = g(x, y)h(t)$ and substitute:

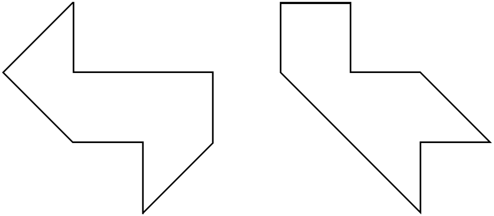
$$\frac{-\Delta g}{g} = -\frac{h''}{h}$$
$$G(x, y) = H(t)$$

Breaking this into two ODEs, we can notice that $G(x, y) = \lambda$ is the exactly the eigenvalue problem and $h(t) = \sin(t\sqrt{\lambda}), \cos(t\sqrt{\lambda})$ so if we assume $u(x, y, 0) = 0$,

$$u(x, y, t) = \sum \sin\left(t\sqrt{\lambda_{n,m}}\right) \sin\left(n\pi x\right) \sin\left(m\pi y\right)$$

Verify Counterexample

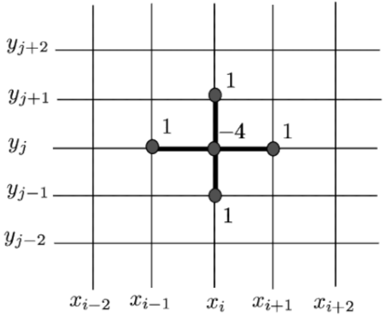
Now that we've seen an easy example, lets find solutions for these regions:



Just kidding! That would be horrible. In fact, these would require completely different techniques since factoring u won't work. In general, it's extremely difficult to solve this problem analytically.

Finite Difference Laplacian

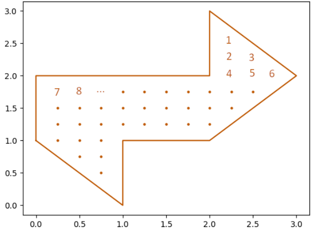
Instead, lets turn to a numerical method for calculating the spectrum. If we discretize our region, we can estimate the Laplacian at a point by looking at its neighbors.



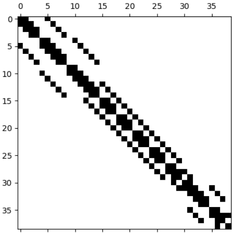
$$\Delta_h \triangleq \frac{1}{h^2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Finite Difference Laplacian

If we index our discretized region, we can start to represent functions on this region as vectors and make a matrix L that approximates the Laplacian on the region as a whole.



$\to L \approx$



Finding the eigenvalues of this matrix will approximate the spectrum of the true operator.

Verify Method

Using a very fine grid, we can get an approximation for the spectrum of the unit square and check it against the values we explicitly calculated.

The first few true eigenvalues

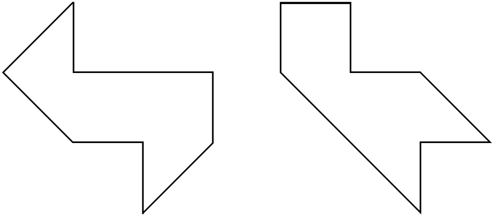
$$\begin{aligned} \lambda &= (n\pi)^2 + (m\pi)^2, n, m \in \mathbb{Z} \\ &= 19.739, 49.348, 49.348, 78.956, 98.696 \end{aligned}$$

The first few eigenvalues the code returns:

$$\lambda = 19.735, 49.314, 49.314, 78.893, 98.533$$

Verify Counterexample

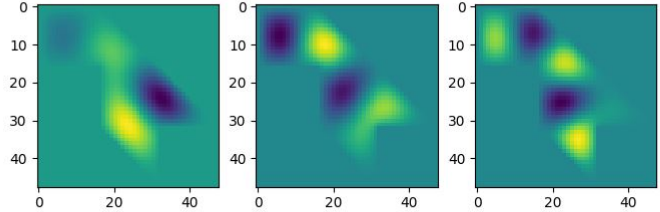
We can use this method with a very fine grid to get a good approximation of the spectra for both regions.



$$\lambda_1 = 10.166, 14.631, 20.718, 26.115, 28.983$$
$$\lambda_2 = 10.166, 14.631, 20.718, 26.115, 28.983$$

Eigenfunctions as a Basis for $\mathsf{L}^2(\Omega)$

Since our method gets the eigenvectors for the L matrix, we can unvectorize them to get approximations of the eigenfunctions.



Eigenfunctions form a basis for the space our solutions lie in, so we can project onto these vectors to get solutions for particular initial conditions.

Questions?

Thanks for listening! Any questions?